Practical Geometric Tomography

Nadav Har'El Shay Gueron
Department of Mathematics, Technion — Israel Institute of Technology

December 11, 1999

Abstract

Tomography deals with reconstruction of density profiles of planar objects given their X-ray projections. Typically, projections in a large number of directions are required for an approximate reconstruction. Geometric Tomography limits the discussion to planar shapes and measurable sets with a constant density. This allows for developing a theory on the possiblility for unique reconstruction from projections in a small, finite, number of directions.

In this paper, we review briefly some important results of Geometric Tomography, and in particular Gardner's algorithm for reconstructing planar convex objects from projections in four directions. We then present a new reconstruction algorithm called *Minverse*. This method allows for flexible "practical reconstruction", using any number of projection directions, and it is not limited to convex objects. We demonstrate the use of the algorithm on various examples, with varying number of projection directions, and successfully reconstruct non-convex star-shaped objects, objects with one or two holes, and disconnected sets.

The experimental results from our implementation of Minverse seem to suggest that unique reconstruction is possible in practice in more cases than previously suggested and proved theoretically.

1 Introduction

Tomography deals with the reconstruction of a slice (a planar section) inside a body, using information from X-rays. Since we are reconstructing slices, we restrict our discussion to planar bodies. One X-ray gives the amount of mass it has passed, and we'll use the term *projection* in a given direction to denote the information coming from radiating the body with all X-rays parallel to the given direction. We shall say that a body can be *uniquely reconstructed* from its projections, among a certain family of bodies, if there is no other body in that family giving the same projections as the given projections.

Most studies of Tomography assume the density function (giving the density at every point) is general, and the result is that projections from an infinite number of directions are needed in order to guarantee a unique reconstruction, or that projections from a finite and large number of directions are needed in order to allow a good approximate reconstruction, under certain assumptions. Another approach is to assume the body has a special symmetry property (e.g., has cylindrical symmetry), and then the density function can be reconstructed from a single projection. These methods of Tomography are used very successfully in medical diagnosis (CAT scans) and for non-destructive evaluation in the industry.

In *Geometric Tomography* (see [6] for a very good and thorogh book on the broader subject) we assume that the body we want to reconstruct from its projections is a *Geometric* set, i.e., one whose density is either 0 or 1 at any point. Because our a-priori assumption on the density function is stronger than that of general Tomography, we expect to get stronger results, and to be able to reconstruct sets from a small number of projections.

It is not possible to reconstruct sets from a projection in a single direction (it is easy to change a set while keeping the projection in one direction unchanged) and so it is reasonable to ask what could be said from projections in two directions. The first result in this area was that of Lorentz from 1949 [10], who talked about the reconstruction of measurable sets among the family of measurable sets. The structure of sets that can be uniquely reconstructed among measurable sets is completely known: Lorentz showed a necessary and sufficient condition for two function to be the two projections from orthogonal directions of some unknown measurable set, and a necessary and sufficient condition for that set to be unique. Later in [9, 4], necessary and sufficient conditions were also found for a given set to have unique reconstruction from two orthogonal projections: A structure called *switching component* was defined, such that a set can be uniquely reconstructed from its two orthogonal projections if and only if it does not contain any switching component. Two other set properties were found to be equivalent to the possibility of unique reconstruction from two orthogonal projections: *inscribability* and *additivity*. These two properties allow for easier testing if a given set can be uniquely reconstructed among measurable sets. Also, a formula was found for the reconstruction, in case a unique reconstruction exists from two orthogonal projections.

Weaker results are known about the reconstruction of measurable sets from three or more projections: it is known [10] that there is no finite set of directions that allows unique reconstruction of *every* measurable set, so we are interested, as before, in conditions on sets that guarantee unique reconstruction given a set of directions. If the definitions of inscribability, additivity and lack of switching component are generalized to any finite set of direction, then these conditions can be seen [6] to be sufficient (but not necessary) for unique reconstruction of the set among the measurable sets, from its projections in those directions. Using these results we can give examples of unique reconstruction: for example, an ellipse is determined by its projections in any set of three directions [6].

If we restrict ourselves to reconstruction among a smaller family of sets, we expect to get different theorems, and to more often be able to uniquely-construct sets from a small number of directions. Hammer, in 1961 [8], raised the question of when can convex planar sets be reconstructed from their projections, among the family of convex planar sets. Since then, many results have been published in this area, although many open questions remain. Some important known results: We know a necessary and sufficient condition for a finite set of directions to allow unique reconstruction of any convex body [7], and in particular that sets of four directions (under some condition) allow unique reconstruction of any convex body. Three or less directions are not enough for reconstructing *every* convex body [7], but it was proved that given two directions, *most* convex bodies (in the sense of category) can be uniquely reconstructed [12]. We also know theorems of different kinds, e.g.: A convex body can be *verified* by projections in three directions (meaning that given a convex body, we can choose three directions such that any other convex body will give a different set of three projections than that of the original body) [5]. On the other hand, there exists a convex polygon (a hexagon) that cannot be verified from its projections in any two directions [5].

Unfortunately, most of the questions of the stability of the reconstruction problems are open, so in many cases there is no theoretical answer to the question of whether it is possible to reconstruct, in practice, certain bodies from certain directions. However, we actually implemented two algorithms for reconstructing sets from their projections that gave good results in a large number of cases, so we conjecture that in the future stronger stability theorems will be proved, proving the usefulness of the aforementioned algorithms.

The first algorithm for practical reconstruction of convex sets from their projections was proposed by Gardner in [6, p. 47]. The algorithm is limited to projections from four directions, and does *not* have full theoretical justification – its convergence and its stability have not been proven. Nevertheless, when implemented properly (see Nadav Har'El's MSc thesis for some details missing from [6]) it does appear to give good practical reconstruction.

2 Minverse

We propose and implement a new reconstruction algorithm which is not limited to a specific number of projections (like Gardner's algorithm needed four projections), and can work not only with convex shapes but also with some generalizations explained below.

Our algorithm, which we named "*Minverse*", solves the reconstruction problem by minimization. The name *Minverse* is a contraction of the words *minimization* and *inverse* (refering to the reconstruction problem as an inverse problem).

The Minverse algorithm tries to reconstruct a star-shaped set (actually, a generalization that also includes the possibility of holes or disconnected sets) given a finite number of X-ray results, when the X-rays are not necessarily directed in four different angles. The idea of the Minverse algorithm is as follows: for a certain guess-body we find the results of the projections by the given X-rays, and we take a norm measuring the distance between the result vector to the required result vector. We try to minimize that norm while changing the body, and declare success when the norm is close enough to zero.

The Minverse algorithm can be used also in cases where there is no uniqueness, and in these cases Minverse finds one of the possible reconstructions. For example, it is known that a square cannot be uniquely reconstructed from two projections, and indeed Minverse found an additional shape — a star-shaped but non-convex shape that wasn't previously mentioned in the literature (see Figure 2). In cases where we do have a theoretical result guaranteeing unique reconstruction, Minverse did reconstruct the body despite the previously mentioned lack of a relevant stability theorem. Also, in many cases where we do not have a theorem guaranteeing or ruling out uniqueness, Minverse did reconstruct the original body, which makes us conjecture that many uniqueness theorems exist that are still waiting to be discovered. Among the shapes we reconstructed using Minverse were convex shapes, non-convex star-shaped shapes, shapes with one or two holes, and even disconnected shapes.

2.1 The design of Minverse

In the introduction above, we gave a very brief survey of results that guarantee the unique reconstruction of some body from its projection in a certain set of directions. For example, we saw that projections in four directions (with some constraint on their choice) uniquely determine any convex body (among the family of convex bodies), and mentioned Gardner's reconstruction algorithm for this special case.

[2, 3, 1] found additional algorithms for special cases: the first two show a heuristic algorithm for the case of a convex body that is symmetric relative to two axes, and the last gives an algorithm for finding (only) round holes in a convex body.

But there are other cases where we know for certain that a unique reconstruction exists. For example, we saw that given only two directions, it is possible to uniquely reconstruct *most* convex bodies among the family of convex bodies, using the projections in these two directions. The problem of classifying the convex bodies that can be reconstructed using two directions is still open, so a reconstruction algorithm would be an insteresting research tool in this respect.

There are possibly other cases where unique reconstruction is possible, but a theorem guaranteeing it is not known yet. E.g., there are no known theorems saying when it is possible to uniquely reconstruct connected, or star-shaped set among the families of connected or star-shaped sets, respectively. If we had a reconstruction tool which can work in these cases too, it might have been possible to come up with conjectures about the uniqueness (or non-uniqueness) in these cases.

This is why we developed a new reconstruction algorithm, which we named *Minverse*, with the following goals in mind:

- It should be possible to reconstruct bodies that are more general than convex bodies: we defined (see below) a generalization of star-shaped polygons, which we called *layered* star-shaped polygons, which also allow for polygons with holes or disconnected polygons.
- The reconstruction tool should get information from any number of projections: we will get a finite number of *ray results*, which are given rays and the *result* of each ray (the amount of mass the ray passed through). These rays are not necessarily directed in four different angles; In principal, each ray can be in a different direction and this tool can be used also for point projections, but we didn't pursue this idea further in this study.
- The algorithm should look for a body that, had we radiated it with the given rays, would give the same ray results as the ones given in advance. The tool would not guarantee unique reconstruction: theoretical results might mean that in certain cases there is no reconstruction at all, or there are reconstructions that are not unique. When there are several possible reconstructions, the algorithm will try to find one of them.

The reconstruction algorithm we developed is based on a minimization algorithm. A brief description of the algorithm is as follows: For some guess body we find the results of the projections of the given rays, and take a norm that measures the distance of the results vector to the vector of the required results. We then try to minimize this norm while changing the guess body, until the norm is close enough to zero.

We called this algorithm, which solves the inverse projection problem (i.e., the reconstruction problem) using minimization, "*Minverse*", from the words *minimization* and *inverse*. To summarize, Minverse is a general method for reconstructing two-dimensional bodies from their projections.

Obviously, for such a reconstruction algorithm (or any other reconstruction algorithm, for that matter) to work, it is not enough to have unique reconstruction. We also need the stability of the reconstruction problem because of the discretization in the algorithm (of the body, the projections, and the directions). Unfortunately, as we have already mentioned, for most cases we do not know that the appropriate stability theorems are true. However, as we shall see below, Minverse gave us the expected reconstructions in many examples, so we conjecture that in the future new stability theorems will be found that explain Minverse's good results.

2.2 Implementing Minverse

In the Minverse algorithm, we have star-shaped polygons with a uniform density. This uniform density should be known in advanced, and so is the number of vertices in the polygon (we will get a better approximation of the body as we increase the number of vertices). Each vertex will be at a predefined angle to the polygon's center, and its distance from the center is the variable that the algorithm will have to find. The center of the star-shaped polygon is also a variable.

By using star-shaped polygons, rather then convex polygons, we allow Minverse to reconstruct a broader class of shapes. However, one should note that in some cases we have a theorem that proves a unique reconstruction of some convex shape within the class of convex shapes, but not within the broader class of star-shaped polygons. When one knows in advance that the sought-after body is convex, it's possible to modify Minverse to look for convex shapes, by penelizing non-convex guesses (see more about penalties below).

Actually, in Minverse we took more general bodies, which we called *layered* star-shaped polygons: layered polygons are constructed from several filled polygons, each with its own uniform density, that are layered one on another — where in a point that is covered by several interiors of polygons the density of the body is taken as the sum of the densities of these polygons. Using this formulation we can have, for example, a unit-density square with a small triangular hole, by taking a square with density $+1$ overlayed with a small triangle with density -1. Using layered polygons we can also represent disconnected bodies. Again, it is necessary to know in advance the number of polygons and their densities, before starting the Minverse algorithm.

To check how close a guess shape is to the shape we're looking for, we need an algorithm to *simulate* a projection, i.e., an algorithm that given a guess shape and a set of rays, gives the density integral that each ray has passed. The projection algorithm works in a straightforward fashion, by finding the intersection between the rays and the different layers of layered-polygon, and then calculating the mass each ray goes through.

The Minverse reconstruction algorithm is basically a minization algorithm. It works by iterating on the following sequence of operations: guessing a shape, determining the distance from the required shape (this will be called the *evaluation function*) and improving the guess. The method of improving the guess is the minimization method itself, and for our tests we used the Downhill Simplex Method described in [11], which is a simple but robust minimization algorithm that does not require derivatives.

We shall now continue to discuss the representation we used for the guess shape (i.e., the *minimization variables*), and its evaluation function: We represent an unknown layered polygon by determining in advance the number of layers, the density of each layer and the number of points in it, and also the angles in which these points are distributed around the origin of the layer. The radii of these points, as well as the origins of the layers, are the variables that represent the guess shape.

The evaluation function is given a set of rays with the desired results, and the required type of guess shape described above. The evaluation function is basically the sum of the squared differences between the result of each ray in the guess shape, and the desired result. However, sometimes we add *penalties* to the evaluation function, to deter the minimization algorithm from straying into illegal or unwanted parts of the parameter space (for example, the radii should all be positive, and negative radii are meaningless). When we find a *bad* guess, we add a *penalty* to the evaluation function, therefore causing the algorithm to stay away from such bad guesses. In our implementation of Minverse we added the following penalties:

1. A penlty on negative radii.
2. A penlty on deviation of the polygon from a circle is added in the *beginning* of the minimization to prevent uncoordinated movement of the seperate points which are all far from their correct positions.

The Downhill Simplex Method we use for the minimization requires an initial guess *simplex*, i.e., an one initial guess and N perturbations on it, where N is the number of variables (we have one variable for each point on a polygon, and two variables for each center of a polygon). We tried various methods of choosing these initial guesses, but the following proved most useful: As an initial guess, all polygons in the layered-polygon are taken as circles, with their centers and radii taken using a very rough guess from the projection data. We take N perturbations on this initial guess, by taking one perturbation for each variable: a radius is perturbed by increasing it, and to a lesser degree the neighboring radii (to prevent very sharp spikes), and a center coordinate is perturbed by increasing it.

The Downhill Simplex Method, as most N-dimensional minimization methods, does not guarantee finding a global minimum (in our case a global minimum would have the value 0, or close to it because of the discretization). The minimization might get stuck in a local minimum or around a guess that is not close enough to a correct solution. This is not only a theoretical concern, and it actually happens in real situations. One practical solution to this problem is to stop the minimization once in a while, and start it over using the best guess we've found so far, hoping that the algorithm will get out of the local "well" it got stuck in. The following procedure gave us good results in Minverse: at the beginning of the minimization process, we stop it relatively often and start over from the last guess. As the algorithm progresses and supposedly tries to converge on a correct solution, we restart it less and less often.

Another refinement of the Minverse algorithm related to restarting the minimization is *recentering*. The fact that the center of each polygon is an ordinary minimization variable is problematic in the following way: When the polygon is close to the correct solution, but the center we chose for it is close to the edge of the polygon, nothing will improve the choice of that center. Moving the center (with the whole polygon) will only make the guess worse. This is why, in a case where our initial guess polygon and its center are far from the solution ones, we often see the center moving "too slow", until we get a polygon whose "center" is on its edge and some of the radii are close to zero, which is problematic because negative radii are not allowed.

Our solution is to *recenter* the polygons whenever the minimization is restarted, i.e., to choose a new center for each polygon (e.g., by averaging the points of the polygon) and then resample the polygon using the new center.

3 Reconstruction results using Minverse

In this section we shall see examples of reconstruction using the Minverse algorithm. In each example, we shall take a given body (a layered polygon), find its projections by a certain set of rays, and than let the Minverse algorithm attempt to reconstruct a body which has these projections. We shall demonstrate how Minverse is able to reconstruct several bodies for which a unique-reconstruction theorem is not known, including non-convex bodies and bodies with holes.

We shall start with an example in which we expect unique reconstruction: a generalized circle with exponent 1.6 ($x^{1.6} + y^{1.6} = 1$), and the two axis directions. Unique reconstruction is proved by the following theorem, which is a generalization of a theorem from [2]:

Theorem 1 *Let A be a set symmetric relative to the y axis (i.e., $(x, y) \in A \iff (-x, y) \in A$), such that its intersection with each line perpendicular to the y axis is a section (this condition is weaker than convexity). Then A is uniquely determined among all measurable sets from its projections in the two axis directions x, y.*

In figure 1 we can see that indeed Minverse was able to reconstruct the generalized circle.

Although we saw a theorem saying that given two directions, *most* convex bodies can be uniquely reconstructed from their projections in these directions, [6] already noted that one of the simplest bodies we can think of, the square, cannot be uniquely reconstructed from its projections in two directions. For any pair of two directions other than the pair of directions of the square's edges, or the pair of directions of the square's diagonals, [6] explains why there is a different parallelogram givin the same projections. In Nadav Har'El's

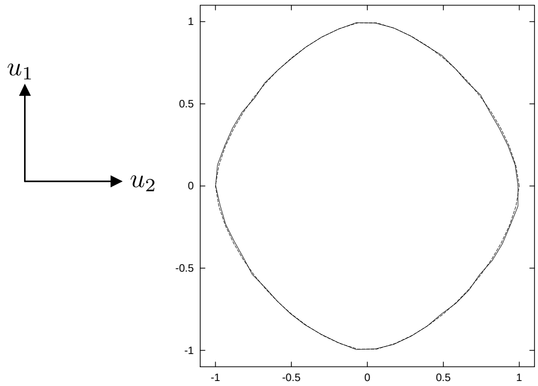

Figure 1: Reconstruction of the generalized circle $x^{1.6} + y^{1.6} = 1$ using Minverse

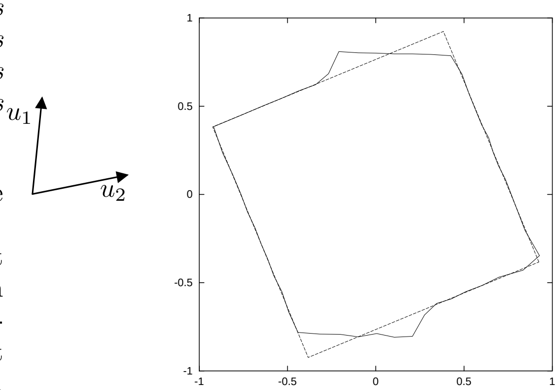

Figure 2: Ambiguous reconstruction of a square using Minverse: Minverse found a different shape with the same projections.

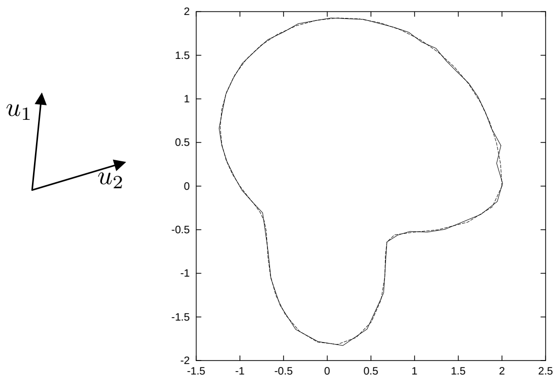

Figure 3: Reconstruction of the “mushroom”

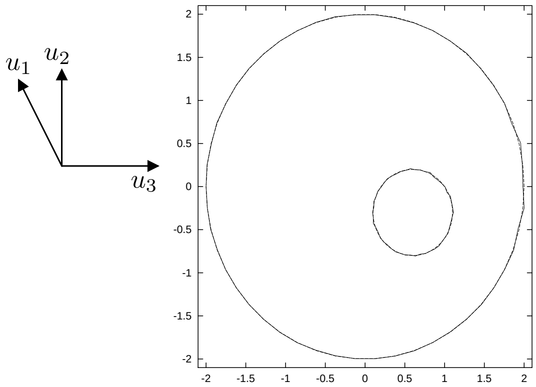

Figure 4: Reconstruction of circle with a circular hole

MSC thesis it is shown why in many choices of two directions, yet another reconstruction that hasn’t been mentioned before is possible, which is star-shaped but not convex. Minverse might find any one of these possible reconstructions, and the actual reconstruction it will find depends on the initial guess. In figure 2 we see how a Minverse reconstruction gave us the aforementioned star-shaped set.

As we said earlier, little is known theoretically about the possibility of unique reconstruction when the given shape is connected (and even star-shaped), but not convex, and Minverse may become an interesting research tool in this area. We’ve already seen above a non-convex but star-shaped shape that has the same projections as a square. In figure 3 we see the Minverse reconstruction of a non-convex “mushroom”-like shape from its projections in two directions. Although we don’t know of a theorem that guarantees unique reconstruction in this case, we can see that indeed Minverse reconstructed the correct shape.

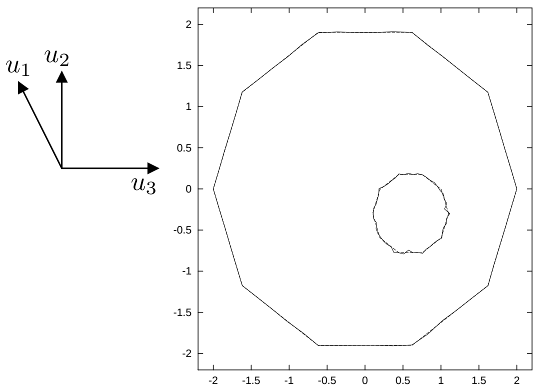

Figure 5: Reconstruction of decagon with a decagonal hole

Our general *layered* polygons allows Minverse to attempt the reconstruction of bodies with holes. Again, the theory in this area is

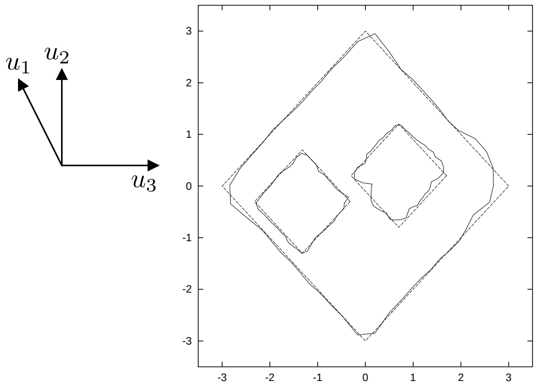

Figure 6: Reconstruction of decagon with two square holes

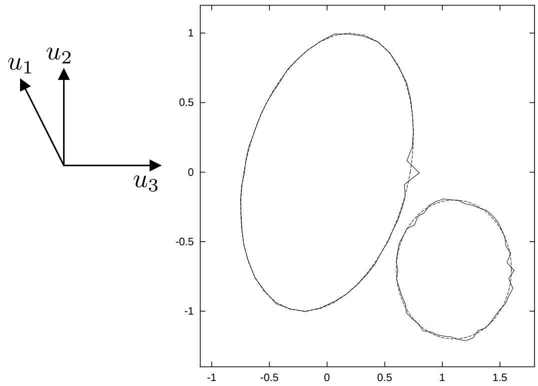

Figure 7: Converged reconstruction of a disconnected shape

largely lacking, making Minverse an interesting research tool. In 4 we see how Minverse reconstructed a circle with a small off-center circlular hole. The first guess was a pair of overlayed circles with radius 1 at the origin. In this reconstruction we used projections from three directions, although we got a similar result by using only two. Another example is the reconstruction of a regular decagon with a hole which is also a regular decagon: see 5. The outer decagon was very accurately reconstructed, while the accuracy of the decagonal hole is lower because of its smaller influence on the evaluation function we are minimazing. 6 shows an example with two holes: in this case the convergence of the Minverse algorithm was very slow, and although as we can see it got quite near the shape we expected, it hasn't quite converged yet (and the evaluation function wasn't near enough to 0).

At this point, we would like to point out that we can think of geometric tomography (the reconstruction of geometric shapes) as a method of reconstruction *holes* in bodies of constant density: suppose that we have an opaque three-dimensional body which has a known uniform density, except a hole whose shape we want to reconstruct. Our methods are planar, so we'll talk about one planar slice of the body. We can measure the outside boundary of the slice with direct methods (X-Rays are not needed for that) and subtract the measured projections from the projections we'd expect *were* the body's extrnal boundary was uniformly filled with material of the given density. The result of the subtraction is exactly the projections that we would have got from an imaginary body which is the hole in the original body filled with material with the given density.

With this view in mind, the reconstruction of a non simply-connected shapes like we did in the above examples does not seem very useful. It seems more useful to try to reconstruct non-connected shapes, e.g., two components representing two holes in a body. For example, in 7 we reconstructed a non-connected shape made of an ellipse and a circle. The initial guess in this reconstruction was two circles of radius 0.5, one centered at the origin and one

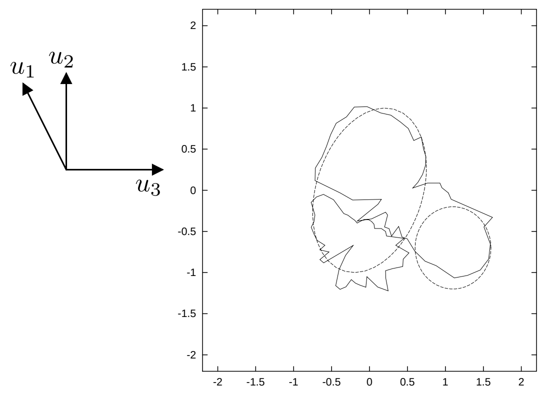

Figure 8: Unconverged reconstruction of a disconnected shape

at $(0, 1)$ (such a crude guess can easily be determined from the projections).

4 Discussion

We developed and implemented a reconstruction tool, Minverse. This tool shows that it is possible in practice to reconstruct star-shaped sets (or even more general sets) from their projections in a small number of directions. These practical results point out that the chances of unique reconstruction are much more optimistic than what it seems from known theoretical results. We conjecture that in the future new theorems will be discovered concerning unique reconstruction and the stability of the reconstruction problem, that will prove why Minverse gives such good results.

Future research on Minverse should include improving the convergence and speed of the algorithm. The current version, written as a research prototype, was not optimized for speed, with typical runs taking anywhere from a few minutes to a few hours on a personal computer. The complexity of the projection simulation subroutine can be lowered using smarter algorithms, and the number of function evaluations can be lowered by using a better minimization algorithm. We should also devise better methods of “escaping” local minimums during the minimization process. For example, when we tried to take for the disconnected shape example an initial guess of two overlayed circles of radius 0.5 at the origin, the Minverse minimization algorithm got stuck at a local minimum, as in Figure 8. This happened because the two polygons have the same density, and start at the same position, so it is ”hard” for them to ”decide” which circle should converge on which connected component.

Acknowledgments

This research was supported by the U.S. - Israel Binational Science Foundation Grant number 97-00400.

N. Har’El acknowledges the support of the Technion Graduate School.

References

[1] G. Bianchi and M. Longinetti. Reconstructing plane sets from projections. *Descrete Comput. Geom.*, 5:223–242, 1990.

[2] Shi Kuo Chang and C. K. Chow. Automatic reconstruction of the heard chanmber from biplane cineangiogram. Technical Report RC 3681, IBM Research, IBM Thomas J. Watson Research Center, Yorktown Heights, New York, November 4 1971.

[3] Shi-Kuo Chang and C. K. Chow. The reconstruction of three-dimensional objects from two orthogonal projections and its application to cardiac cineangiography. *IEEE Trans. Comput.*, 22:18–28, 1973.

[4] P. C. Fishburn, J. C. Lagarias, J. A. Reeds, and L. A. Shepp. Sets uniquely determined by projections on axes i. continuous case. *SIAM J. Appl. Math.*, 50(1):288–306, February 1990.

[5] R. J. Gardner. Symmetrals and x-rays of planar convex bodies. *Arch. Math.*, 41:183–189, 1983.

[6] R. J. Gardner. *Geometric Tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1995.

[7] R. J. Gardner and P. McMullen. On hammer's x-ray problem. *J. London Math. Soc.*, 21(2):171–175, 1980.

[8] P.C. Hammer. Problem 2. In *Proceedings of Symposia in Pure Mathematics*, volume VII: Convexity, pages 498–499, Providence, RI, 1963. American Mathematical Society.

[9] A. Kuba and A. Volčič. Characterisation of measurable plane sets which are reconstructable from their two projections. *Inverse Problems*, 4:513–5–27, 1988.

[10] G. G. Lorentz. A problem of plane measure. *Amer. J. Math*, 71:417–426, 1949.

[11] William H. Press, William T. Vetterling, Saul A. Teukolsky, and Brian P. Flannery. *Numerical Recipes in C*. Cambridge University Press, second edition edition, 1994.

[12] A. Volčič and T. Zamfirescu. Ghosts are scarce. *J. London Math. Soc. (2)*, 40:171–178, 1989.