CURVATURE OF CURVES AND SURFACES – A PARABOLIC APPROACH

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Abstract. Parabolas and paraboloids are used to introduce curvature, both qualitatively and quantitatively.

1. Introduction.

Most of the standard differential geometry textbooks recognize the *osculating paraboloid* as a useful interpretation of the second fundamental form of a surface. This interpretation leads to some qualitative conclusions about the shape of the surface in consideration [9, pp.87,91]. Two extensions are usually ignored:

(a) The osculating paraboloid has a planar counterpart, the *osculating parabola* to a curve in $\mathbb{R}^2$. This fact is mentioned only in passing as an exercise [5, p.26] [4, p.47], if at all.

(b) The fact that the osculating paraboloid may be used to produce concrete *quantitative* results, such as formulas for the *Gaussian curvature* and the mean curvature of a surface, or even *Meusnier's formula* for the curvature of skew planar sections.

With these observations in mind, we introduce a unified approach to curvature in two or three dimensions using *quadratic approximation* as the fundamental concept. Geometrically, we *define* curvature of a planar curve as the reciprocal of the semi-latus-rectum of the osculating parabola, instead of considering the familiar rate of change in direction of the tangent. The latter approach is easily visualized via the osculating circle, which, however, does not lend itself to generalization to higher dimensions, since surfaces have no osculating spheres (at non-umbilical points, of course). In contrast, osculating parabolas to curves in $\mathbb{R}^2$ are easily and naturally generalized to osculating paraboloids to hypersurfaces in $\mathbb{R}^n$. It should be admitted that by rejecting the use of parametric representation and differential forms we lose much of the intrinsic geometry of surfaces. As compensation, we are given a comprehensive understanding of the notions of curvature of curves and surfaces, both qualitatively and quantitatively, without using any machinery more powerful than Taylor's approximation formula in its simplest form (i.e. with "asymptotic" remainder [6, p.230] [2, p.71]).

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2. Parabolas.

Consider the one-parameter family of parabolas, with vertex at the origin and the y-axis as its axis of symmetry. The cartesian equation of such a "canonical parabola" is $x^2 = 2py$, or $2y = \frac{1}{p}x^2$, with p, the semi latus-rectum, being the parameter. If several parabolas are graphed on a common chart, one notices that as $|p|$ is increased, the "flatness" of the parabola at its vertex increases, while its "curvature" decreases. This makes the following definition plausible:

Definition 2.1. The *(numerical) curvature* of a parabola *at its vertex* is $\left|\frac{1}{p}\right|$.

The above definition includes the vanishing of the curvature of a straight line as a special case, i.e. a degenerate parabola with $\frac{1}{p} = 0$. Furthermore, one notices that a canonical parabola is concave upward for $p > 0$, and downward for $p < 0$. Thus the sign of p (or $\frac{1}{p}$) signifies the direction of concavity. We define:

Definition 2.2. The *(signed) curvature* κ of a parabola *at its vertex* is $\kappa = \frac{1}{p}$.

The choice of positive direction of concavity is of course arbitrary, but once it is made, the sign of κ enables us to distinguish between concave and convex.

3. Curvature of Plane Curves.

Given a general plane curve, we wish to define its curvature as the curvature of its closest parabolic approximation. The most enlightening example is that of the circle:

It is well known that a beam of light, coming from infinity parallel to the axis of symmetry and reflected by a paraboloidal mirror, converges at the focus of the mirror. The same property holds approximately for a narrow beam reflected by a circular (cylindrical, spherical) mirror, with the focus located half-way between the center of the circle and the point of reflection. This means that a circular arc with radius R can be approximated by a parabolic arc with focal length $\frac{p}{2} = \frac{R}{2}$, or semi latus-rectum $p = R$ (see Figure 1).

The last observation can also be verified directly, by finding the quadratic approximation near the origin to the ordinate of a circle with center at $(0, R)$ and radius R. From the equation $x^2 + (y - R)^2 = R^2$, we obtain, for the lower semi-circle,

$$\begin{aligned} y &= R - \sqrt{R^2 - x^2} \\ &= R - R\sqrt{1 - \frac{x^2}{R^2}} \\ &= R - R\left(1 - \frac{x^2}{2R^2} + o(x^2)\right) \\ &= \frac{1}{2R}x^2 + o(x^2). \end{aligned}$$

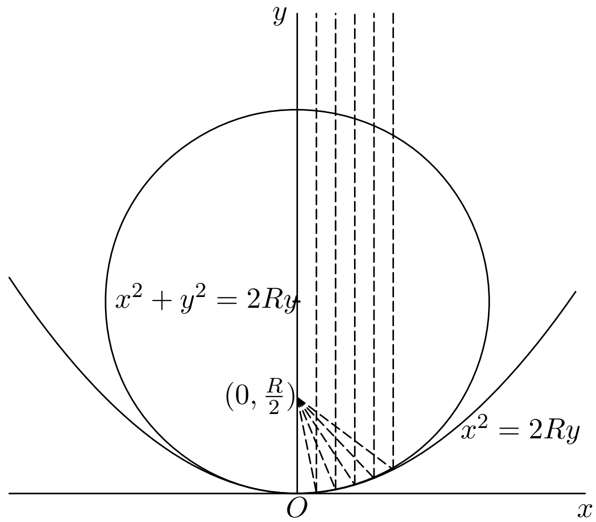

FIGURE 1. Parabolic and circular mirrors

Thus,the circular arc is approximated near $(0,0)$ by the parabola $2y = \frac{1}{R}x^2$, with curvature $\kappa = \frac{1}{R}$ at the vertex.

Definition 3.1.

i. A parabola which has contact of order ≥ 2 at its vertex P with a curve C is called the *osculating parabola* of C at P.
ii. The *curvature* of C at P is the curvature at the vertex of the osculating parabola of C at P (assuming the existence of the latter).

Recall: Two curves have *contact of order* $\geq k$ at a point whose abscissa is x_0 if [6, p. 297] the difference of their ordinates at the point whose abscissa is $x_0 + h$ vanishes to a higher order than the k-th power of h. Geometrically, this implies that the osculating parabola (if it exists at all) separates the family of parabolas with vertex at P and symmetry axis along the normal to C at P (i.e. parabolas having contact of order ≥ 1 with C at P) into two sub-families, each consisting of parabolas which lie on one side of C in some neighborhood of P. In this sense, the osculating parabola is the best approximation to C among these parabolas. Note that the above definition can also be applied to curves in $\mathbb{R}^3$.

As an example, we use Definition 3.1 to prove the existence of the osculating parabola and to compute the curvature of a curve C with the cartesian equation $y = f(x)$, at a point $P(x_0, y_0)$, assuming that f is twice differentiable at x_0.

Consider a new coordinate system (ξ, η), whose origin is at the given point P, ξ-axis is the tangent line at P, and η-axis is the upward normal (see Figure 2). Vectorially, the new axes have the positive directions given by $(1, f'(x_0))$ and $(-f'(x_0), 1)$, respectively. Let $Q(x, f(x))$ be another point on C, then its η-coordinate is its (signed) distance from the tangent line at P, namely,

$$\eta = \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\sqrt{1 + f'(x_0)^2}} \tag{3.1}$$

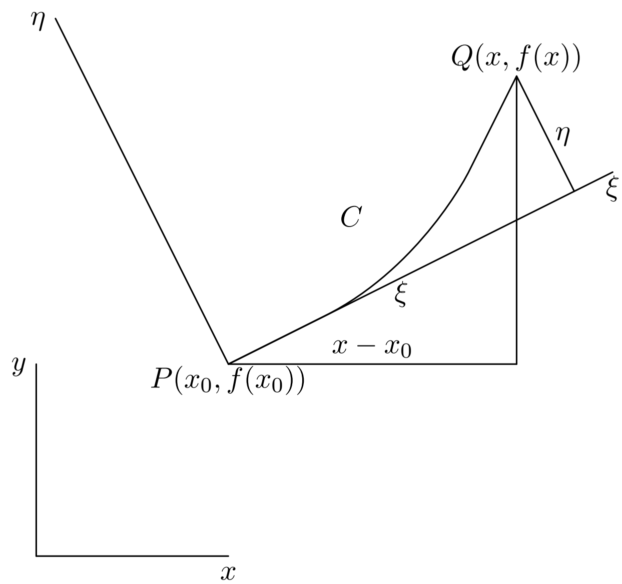

FIGURE 2. Local coordinate system

(positive for Q above the tangent). This implies, by the quadric approximation of f from Taylor's formula for Q near P (assuming only that $f''(x_0)$ exists), that: (assuming only that $f''(x_0)$ exists):

$$\eta = \frac{f''(x_0)}{2\sqrt{1+f'(x_0)^2}}(x-x_0)^2 + o((x-x_0)^2). \tag{3.2}$$

On the other hand, we have

$$x - x_0 = \frac{1}{\sqrt{1+f'(x_0)^2}}\xi - \frac{f'(x_0)}{\sqrt{1+f'(x_0)^2}}\eta, \tag{3.3}$$

a formula which can be deduced by considering similar triangles or by equating the first components of the vector equation:

$$(x - x_0, y - y_0) = \frac{(1, f'(x_0))}{\sqrt{1+f'(x_0)^2}}\xi + \frac{(-f'(x_0), 1)}{\sqrt{1+f'(x_0)^2}}\eta. \tag{3.4}$$

Substituting the value of η given by (3.2) into (3.3) we obtain

$$x - x_0 = \frac{1}{\sqrt{1+f'(x_0)^2}}\xi + o(x - x_0)$$

and hence, by the inverse function theorem,

$$x - x_0 = \frac{1}{\sqrt{1+f'(x_0)^2}}\xi + o(\xi). \tag{3.5}$$

Thus, from (3.2) and (3.5), we obtain

$$2\eta = \frac{f''(x_0)}{(1+f'(x_0)^2)^{3/2}}\xi^2 + o(\xi^2). \tag{3.6}$$

We conclude that the osculating parabola is given by $2\eta = \kappa\xi^2$, where $\kappa = f''(x_0)(1+f'(x_0)^2)^{-3/2}$ is the curvature of C at $P(x_0, f(x_0))$ according to Definition 3.1 - agreeing with the standard definition - positive if C lies locally above the tangent, i.e. if C is concave upward.

4. Paraboloids.

In the same way we determined the curvature of a plane curve by comparing it with its osculating parabola, we wish to compare surfaces in a 3-dimensional space with paraboloids (i.e. non-central quadrics [8, p.99]). Consider the family of paraboloids, each having its vertex at the origin and the z-axis as the normal at the vertex. (Thus, the z-axis is always an axis of symmetry.) This is a 3-parameter family given by the cartesian equation $2z = Ax^2 + 2Bxy + Cy^2$. We classify paraboloids according to the type of their sections with horizontal planes ($z =$ const.). These sections are all similar to the (pair of) conic(s) $Ax^2 + 2Bxy + Cy^2 = \pm 1$, called *Dupin's indicatrix* [3, p.363] of the paraboloid. Thus, we get elliptic paraboloids if $AC - B^2 > 0$, hyperbolic paraboloids if $AC - B^2 < 0$, and parabolic cylinders (considered as non-central quadrics of parabolic type) if $AC - B^2 = 0$. It is well known (and easily shown) that the discriminant $K = AC - B^2$ is invariant under rotations of the xy-plane, as is $H = (A + C)/2$. Conversely, K and H define the indicatrix, and hence the paraboloid, uniquely up to a rotation. Note that K is independent of the orientation of the coordinate system (i.e. which way is "up"), while H changes its sign when the orientation is reversed (compare Section 2).

Definition 4.1. The *Gaussian curvature*, K, and the *mean curvature*, H, of the paraboloid $2z = Ax^2 + 2Bxy + Cy^2$ at its vertex are $K = AC - B^2$ and $H = (A + C)/2$, respectively.

We now interpret K and H using the curvature (as defined in Section 2) of normal sections of the paraboloid at its vertex. Recall that a *section* of a surface S is the intersection of S and a plane; a *normal section* of S at a point P is the section of S by a *normal plane*, i.e. a plane containing the normal to S at P. By a suitable rotation of the xy plane, we may assume that B is zero, so that the x-axis and y-axis are the principal axes of the Dupin's indicatrix. We wish to find the normal sections of the paraboloid $2z = Ax^2 + Cy^2$ at the origin, by a plane making an angle θ with the x-axis.

Using cylindrical coordinates, we set $x = r\cos\theta$, $y = r\sin\theta$, and obtain

$$2z = (A\cos^2\theta + C\sin^2\theta)r^2.$$

Thus the normal section is a parabola, with curvature

$$\kappa_n = A\cos^2\theta + C\sin^2\theta = \frac{A+C}{2} + \frac{A-C}{2}\cos 2\theta$$

at its vertex. Assuming, for the sake of definiteness, that $A \geq C$, we find

$$\max \kappa_n = \kappa_n|_{\theta=0} = \frac{A+C}{2} + \frac{A-C}{2} = A,$$
$$\min \kappa_n = \kappa_n|_{\theta=\pi/2} = \frac{A+C}{2} - \frac{A-C}{2} = C.$$

Since $AC = K$ and $A + C = 2H$, we conclude:

Theorem 4.1 (Euler's formula for paraboloids)**.** *The intersection of a paraboloid with a a normal plane at its vertex P is a parabola, whose curvature at P is given by*

$$\kappa_n = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta,$$

where κ_1 *and* κ_2 *are the roots of the equation* $\kappa^2 - 2H\kappa + K = 0$*, and* θ *is the angle between the given plane and the plane for which* κ_n *attains its maximum (if* $\kappa_1 \geq \kappa_2$*, minimum otherwise).*

The quantity κ_n is called the *normal curvature* of the paraboloid in the direction of the tangent to the given normal section at the vertex, while its extreme values, κ_1 and κ_2, are the *principal curvatures.* They are attained in the directions of the principal axes of Dupin's indicatrix, whence these directions are called *principal directions.*

To complete the discussion, we investigate the nature of the non-normal sections of a paraboloid, i.e. sections with planes through the vertex, making angles ϕ $(0 < |\cos\phi| < 1)$ with the z-axis. Assume, without loss of generality, that the paraboloid $2z = Ax^2 + 2Bxy + Cy^2$ is cut by a plane containing the x-axis. Using cartesian coordinates (ξ, η) in this plane, we set $x = \xi$, $y = \eta \sin\phi$, and $z = \eta\cos\phi$, obtaining

$$2\eta\cos\phi = A\xi^2 + 2B\xi\eta\sin\phi + C\eta^2\sin^2\phi. \tag{4.1}$$

This is a conic whose type is determined by the sign of the discriminant $(AC - B^2)\sin^2\phi$, i.e. the type of the paraboloid. Furthermore, since this conic is tangent to the ξ-axis, $\eta = o(\xi)$ and, from (4.1),

$$2\eta\cos\phi = A\xi^2 + o(\xi^2),$$

a curve which has curvature $\kappa = A/\cos\phi$ at $\xi = 0$. On the other hand, the normal section having the same tangent at the vertex is given by $y = 0$, $2z = Ax^2$, with normal curvature $\kappa_n = A$. Thus we have:

Theorem 4.2 (Meusnier's formula for paraboloids)**.** *A plane through the vertex* P *of a paraboloid, making an angle* ϕ *(such that* $0 < |\cos\phi| < 1$*) with its normal axis, cuts the paraboloid in a conic (having the same type as the paraboloid) whose curvature at* P *is given by*

$$\kappa = \frac{\kappa_n}{\cos\phi}$$

where κ_n *is the normal curvature of the paraboloid in the direction of the tangent to the conic at* P*.*

Remark . A positive (negative) cosine applies when the positive direction of concavity in the given plane makes an acute (respectively, obtuse) angle with the upward pointing normal.

5. Curvature of Surfaces.

We now use the results of the last section to investigate general surfaces.

Definition 5.1.

i. A paraboloid which has contact of order ≥ 2 at its vertex P with a surface S is called the *osculating paraboloid* of S at P.
ii. The *Gaussian curvature*, *mean curvature*, *Dupin's indicatrix*, *normal curvature*, etc. of S at P are respectively, the Gaussian curvature, mean curvature, Dupin's indicatrix, normal curvature etc. at the vertex of the osculating paraboloid (assuming its existence).
iii. P is an *elliptic*, *hyperbolic* or *parabolic* point of S if the osculating paraboloid is elliptic, hyperbolic, or a parabolic cylinder, respectively.

The derivation of the equation of the osculating paraboloid of a twice differentiable surface, in a manner analogous to that of Section 3, is differed to Section 7, because it is more technical. At this point we notice that Theorems 4.1 and 4.2 immediately imply the following two classical theorems:

Theorem 5.1 (Euler). *The intersection of a surface with a normal plane at a point P is a curve, whose curvature at P is given by*

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

where κ_1 and κ_2 are the roots of the equation $\kappa^2 - 2H\kappa + K = 0$, and θ is the angle between the given plane and the plane for which κ_n attains its maximum (if $\kappa_1 \geq \kappa_2$, minimum otherwise).

Theorem 5.2 (Meusnier). *A plane through a point P of a surface S, making an angle ϕ (such that $\cos\phi \neq 0$) with the normal at P, cuts S in a curve whose curvature at P is given by*

$$\kappa = \frac{\kappa_n}{\cos\phi},$$

where κ_n is the normal curvature of S in the direction of the tangent to the given section at P.

Both theorems follow from the fact that the second order contact of S with its osculating paraboloid at P implies the same order of contact for the plane sections of S with the corresponding sections of the paraboloid. We also note that the foregoing treatment of surfaces in $\mathbb{R}^3$ extends to hypersurfaces in $\mathbb{R}^n$.

6. Applications.

As an application of the above theory, we easily compute the Gaussian and mean curvature of a surface of revolution S, obtained (in cylindrical coordinates) by rotating the curve $r = f(z)$ about the z-axis. Since any plane through the axis of revolution is a plane of symmetry of S, we deduce that the osculating paraboloid at a given point $P(r_0, \theta_0, z_0)$ of S is symmetric about the plane containing P and the z-axis. This is a normal plane which cuts S in the meridian $\theta = \theta_0$, with curvature

$$\kappa_1 = \frac{f''(z_0)}{(1 + f'(z_0)^2)^{3/2}} \tag{6.1}$$

(see Section 3). The above symmetry implies this is one of the principal curvatures, the other one being the normal curvature in the perpendicular direction, that is tangent to the parallel $z = z_0$ at P. Since the parallel is a circle with curvature $1/r_0$, and its radius makes an obtuse angle ϕ with the outward normal at P (see Figure 3) such that $\tan\phi = -|f'(z_0)|$, we obtain the value

$$\kappa_2 = \frac{\cos\phi}{r_0} = -\frac{1}{f(z_0)\sqrt{1 + f'(z_0)^2}} \tag{6.2}$$

for the corresponding normal curvature by using Meusnier's formula (which applies since $\cos\phi \neq 0$. Note that $|\kappa_2|$ is the reciprocal of the length of the normal segment PN in Figure 3). From (6.1) and (6.2) we get

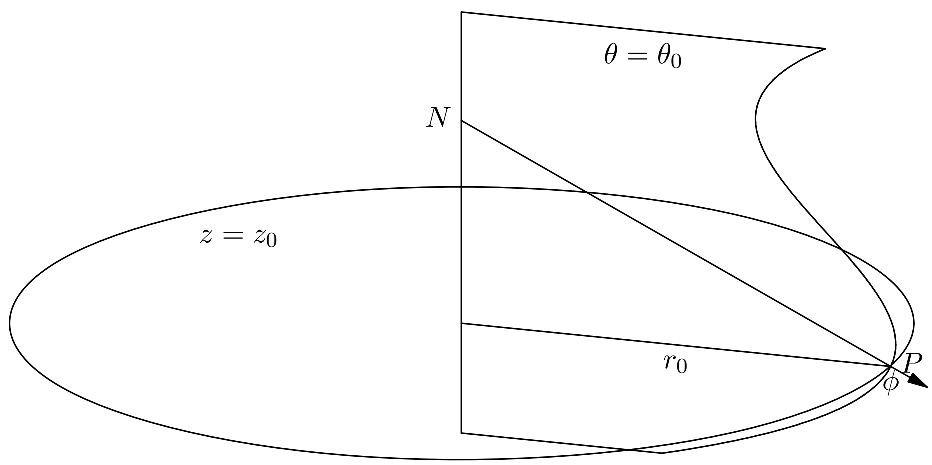

FIGURE 3. Surface of revolution

$$K = \kappa_1\kappa_2 = -\frac{f''(z_0)}{f(z_0)(1+f'(z_0)^2)^2},$$
$$H = \frac{\kappa_1+\kappa_2}{2} = \frac{f(z_0)f''(z_0) - f'(z_0)^2 - 1}{2f(z_0)(1+f'(z_0)^2)^{3/2}}, \tag{6.3}$$

the usual formulas for the Gaussian curvature and the mean curvature, respectively.

Note that as an extra bonus we have shown that the meridians and parallels of S are *principal curves* or *lines of curvature*, that is, curves whose tangents are always in principal directions.

As a specific application of formulas (6.1) to (6.3) consider the *catenoid* $r = a\cosh z/a$ which satisfies the differential equations $r/a = (1 + r'^2)^{1/2} = ar''$ and thus has the principal curvatures $\kappa_1 = a/r^2$ along the meridians (catenaries) and $\kappa_2 = -a/r^2$ along the parallels. Consequently $K = -a^2/r^4$ and $H = 0$ everywhere. (Note that $\kappa_1 = -\kappa_2$ is due to the characterization of the catenary as the only curve whose curvature at P is equal to the reciprocal of the normal segment PN).

The theory of asymptotic directions and minimal surfaces provides a second application of the osculating paraboloid. A direction of vanishing normal curvature on a surface is called *asymptotic*. Euler's Theorem 4.1 implies the existence at a hyperbolic point of two asymptotic directions. This is due to the fact that the sections of the osculating hyperbolic paraboloid in the directions of the asymptotes of Dupin's indicatrix are straight lines (whence the name "asymptotic"). In particular the asymptotes are mutually perpendicular if and only if the indicatrix is an equilateral hyperbola. In other words, the asymptotic directions are orthogonal if and only if the principal curvatures are equal in absolute value, i.e. $H = (\kappa_1 + \kappa_2)/2 = 0$. A surface with identically zero mean curvature is said to be *minimal*, as it can be shown to minimize surface area (locally, with respect to a fixed boundary [5, p.219]). One example is the catenoid (mentioned above), which is the only minimal surface of revolution (except the plane [4, p.202]).

Another instructive example is provided by the *helicoid*, the ruled surface given by the cylindrical equation $z = a\theta$. Recall that a *ruled surface* is a

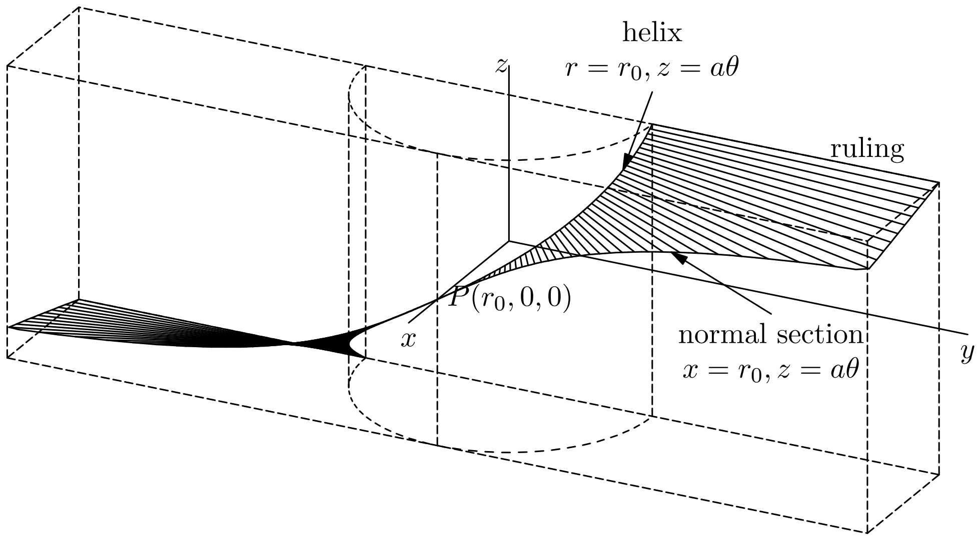

FIGURE 4. The helicoid $z = a\theta$

surface swept out by a line moving along a curve (the z-axis, in this case); the various positions of the line are called *rulings*. To investigate the nature of the surface at a given point $P(r_0, \theta_0, z_0)$, we observe that the surface is symmetric about the ruling $\theta = \theta_0, z = z_0$ through P (i.e. symmetric with respect to a half-turn $\theta \to 2\theta_0 - \theta, z \to 2z_0 - z$). The osculating paraboloid at P inherits this symmetry, and since the ruling is not the normal in its vertex, the paraboloid is necessarily an equilateral hyperbolic paraboloid (check that any other paraboloid has only the normal as an axis of symmetry), proving that the helicoid is minimal! For added clarity, we describe two orthogonal sections of the helicoid with vanishing normal curvature at P, as in Figure 4 (where we put $\theta_0 = 0$ without any loss of generality). First, any plane containing the ruling (the x-axis in Figure 4) through P, the normal plane included, cuts the surface in a straight line. Second, the plane ($x = r_0$ in Figure 4) perpendicular to the ruling at P yields a normal section of the helicoid which is symmetric about P, and the symmetry implies that P is an inflection point, in other words the osculating parabola of the section at P is a straight line. But the line is also tangent to the helix $z = a\theta, r = r_0$. Thus the helices on the helicoid have the property (shared obviously by the rulings) that their tangents always lie in an asymptotic direction. Curves with this property are called *asymptotic curves*.

In addition to the principal curves and asymptotic curves, there is a third, even more important kind of a curve. One can easily verify that Meusnier's formula holds for the curvature of any (not necessarily planar) curve on the surface, as long as the plane of its osculating parabola makes an angle ϕ with the normal to the surface. Thus, we have $|\kappa| \geq |\kappa_n|$, with equality if and only if $\phi = 0$, in which case we get the "straightest" curve possible [7, p.221]. This motivates the following:

Definition 6.1. A curve C on a surface S is called a *geodesic* if, for every point P on C, the plane of the osculating parabola to C at P contains the normal to S at P (or if the osculating parabola to C at P degenerates to a line, in which case the normal plane containing this line may be chosen).

The great importance of geodesics is the fact that in addition to being the straightest they are also the "shortest" curves on the surface, and hence they are a generalization to lines in the plane [7, p.222]. Geodesics minimize distance in two senses: Locally, it may be shown [5, p.265] that if two points on a geodesic are close enough, then the geodesic segment between them is the shortest curve joining these points. Globally, any two points on a complete surface (intuitively, a surface without holes or edges) may be joined by a geodesic, which is the shortest curve between the points (Hopf-Rinow Theorem [5, p.285]).

Great circles on a sphere, being normal sections, are geodesics. Two antipodal (diametrically opposite) points on the sphere may be joined by infinitely many great semicircles, each of which minimizes the distance between the two points. Given two non-antipodal points, one can draw a unique great circle through the points and get two unequal geodesic segments joining them. Of course, only the shorter one minimizes distance globally.

As mentioned above, the meridians of a surface of revolutions are normal sections, and hence geodesics. A parallel $z = z_0$ in a surface of revolution is a geodesic if is a normal section, i.e. if $\phi = \pi$ or $f'(z_0) = 0$. Hence, parallels of extreme radius, called *equators*, are geodesics. Such an equator is for instance the "waist", $z = 0$ and $r = a$ of the catenoid. Finally, since the osculating parabola to a line lying on a surface is just that line, all the rulings of a ruled surface are geodesics.

Geodesics were introduced as the straightest curves on a surface S. As the plane of the osculating parabola at a point P contains the normal to S at P, the orthogonal projection of the geodesic on the tangent plane to S at P has a line for its osculating parabola, hence its curvature vanishes at P. (This is because the osculating parabola of the projection is the projection of the osculating parabola!) Thus, we define the *geodesic curvature* κ_g of a curve C at a point P on S as the curvature at P of the orthogonal projection of C on the tangent plane to S at P. It measures the deviation of C from being a geodesic, and is the intrinsic curvature of C as a subset of S (i.e. the curvature of C as seen by two dimensional inhabitants of S). Let ϕ be as above, we have

$$\kappa_g = \kappa \sin \phi \tag{6.4}$$

(as the focus a projection of a parabola is the projection of its focus). Combining (6.4) with Meusnier's formula

$$\kappa_n = \kappa \cos \phi$$

we get immediately

$$\kappa^2 = \kappa_n^2 + \kappa_g^2. \tag{6.5}$$

Being straightest curves, geodesics have a useful kinematic interpretation: The orbit on a surface S which is traced by a particle which has no acceleration component tangential to S is a geodesic. As the osculating parabola has a contact of order ≥ 2 with the curve, a particle moving along an orbit C has the same velocity and acceleration vectors at a point P as if it moved along the osculating parabola to C at P with the same speed ds/dt and tangential acceleration d^2s/dt^2 at P. In particular, if it moves along C at a constant speed, one deduces it accelerates only in the direction of the normal of the osculating parabola (called the principal normal of C) at P and in the case C is a geodesic this normal coincides with the normal of S at P, as stated.

Kinematic considerations are most helpful in studying geodesics on a surface of revolution S [1, p.153]. Let C be such a geodesic. As the normal to S always meets the z-axis, a constant-speed motion along C will project orthogonally on a central motion in the xy-plane, i.e. a motion whose acceleration vector always points to the origin. Such "planetary motion" obeys Kepler's Law of Equal Areas, which may be written, in polar coordinates,

$$\text{time derivative of swept area} = \frac{r^2}{2}\frac{d\theta}{dt} = \text{constant.}$$

If we return to C, we can replace $r\, d\theta/dt$, the velocity component tangential to the parallel, by $ds/dt \cos\psi$, where ψ is the angle C forms with the parallel at P. Hence,

$$\frac{r^2}{2}\frac{d\theta}{dt} = \frac{r}{2}\frac{ds}{dt}\cos\psi,$$

and as ds/dt is constant we get Clairaut's formula [5, p.267]

$$r\cos\psi = c \tag{6.6}$$

with c constant along C.

Formula (6.6) may be used to find the equation of any geodesic on a surface of revolution S, given as initial data a point r_0 and a direction ψ_0. We refer the reader to [4, p.257] for further details. We just mention here that for $\psi_0 = \pi/2$ we have

$$c = r_0 \cos\psi_0 = 0$$

from which we obtain the meridian $\psi \equiv \pi/2$, and if $\psi_0 = 0$ and the parallel $r = r_0$ is a geodesic, we get that parallel. In any other case, the geodesic satisfies

$$r \geq c = r_0 \cos\psi_0,$$

which implies it always stays on one side of the parallel $r = c$ and if they meet, they touch each other without crossing (as in this case $\cos\psi = c/r = 1$ we have $\psi = 0$ and as the parallel $r = c$ is not a geodesic, it cannot be an equator, which implies $r < c$ on one of its sides). In Figure 5 we sketch four typical geodesics on the catenoid $r = a\cosh z/a$, corresponding to four values of c:

A. For $c = 0$, a meridian.
B. For $0 < c < a$, a geodesic which crosses all the parallels.
C. For $c = a$, the equator $r = a$.
D. For $c > a$, a geodesic which lies on one side of a parallel $r = c$.

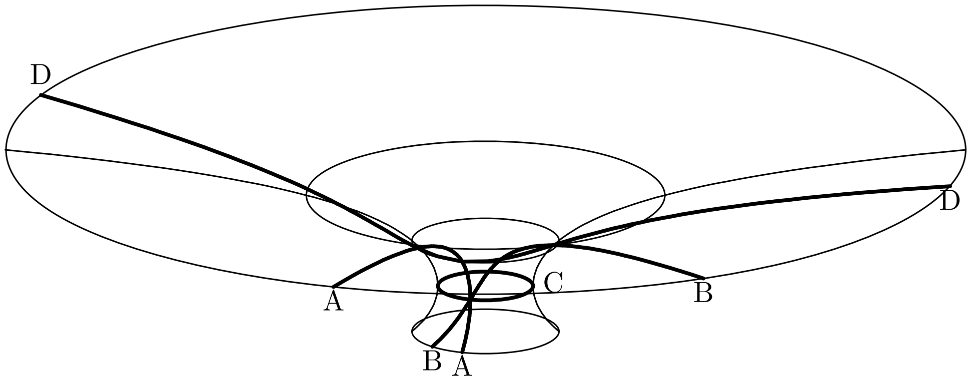

FIGURE 5. Typical geodesics on the catenoid

All these geodesics are given by the elliptic integral

$$\theta = \pm c \int \frac{dr}{\sqrt{(r^2 - a^2)(r^2 - c^2)}}.$$

7. Curvature Formulas.

In this section we indicate how the procedure described in Section 3, formulas (3.1) to (3.6), generalizes to yield the quadratic approximation of a surface S given in *Monge's form* $z = f(x, y)$ [3, p.343] in the neighborhood of a point $P(x_0, y_0, f(x_0, y_0))$, assuming that f is twice differentiable at (x_0, y_0) (in the sense of [2, p.58]). We then deduce the formulas for the Gaussian curvature and the mean curvature of S at P.

Consider a new cartesian coordinate system (ξ, η, ζ), whose origin is P, $\xi\eta$-plane is the tangent plane of S at P, and positive ζ-axis is the upward normal. In fact, we chose the ξ-axis to be parallel to the xz-plane so that the new axes have positive directions given by the vectors $(1, 0, f_x)$, $(-f_x f_y, 1 + f_x^2, f_y)$ and $(-f_x, -f_y, 1)$ respectively (the η-direction is obtained by taking the vector product of the ζ- and ξ-directions), where all the derivatives are computed at (x_0, y_0). By analogy with (3.1), we have for the ζ-coordinate of a point $Q(x, y, f(x, y))$,

$$\zeta = \frac{f(x, y) - f(x_0, y_0) - f_x \cdot (x - x_0) - f_y \cdot (y - y_0)}{\sqrt{1 + f_x^2 + f_y^2}} \tag{7.1}$$

and, using the quadratic approximation to f from Taylor's formula (assuming f is twice differentiable at (x_0, y_0)),

$$\zeta = \frac{f_{xx} \cdot (x - x_0)^2 + 2f_{xy} \cdot (x - x_0)(y - y_0) + f_{yy} \cdot (y - y_0)^2}{2\sqrt{1 + f_x^2 + f_y^2}} + o((x - x_0)^2 + (y - y_0)^2). \tag{7.2}$$

On the other hand, we have

$$(7.3)\quad \begin{aligned} x - x_0 &= \frac{1}{\sqrt{1+f_x^2}}\xi - \frac{f_x f_y}{\sqrt{(1+f_x^2+f_y^2)(1+f_x^2)}}\eta - \frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\zeta, \\ y - y_0 &= \frac{1+f_x^2}{\sqrt{(1+f_x^2+f_y^2)(1+f_x^2)}}\eta - \frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\zeta, \end{aligned}$$

formulas which can be deduced by equating components of 3-dimensional analog to the vector equation (3.4):

$$(7.4)\quad (x - x_0, y - y_0, z - z_0) = \frac{(1, 0, f_x)}{\sqrt{1+f_x^2}}\xi + \frac{(-f_x f_y, 1+f_x^2, f_y)}{\sqrt{(1+f_x^2)(1+f_x^2+f_y^2)}}\eta + \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}\zeta.$$

But, since ζ is quadratic is $x - x_0$ and $y - y_0$, and therefore quadratic in ξ and η by the inverse function theorem, it follows that:

$$(7.5)\quad \begin{aligned} x - x_0 &= \frac{1}{\sqrt{1+f_x^2}}\xi - \frac{f_x f_y}{\sqrt{(1+f_x^2+f_y^2)(1+f_x^2)}}\eta + o(|\xi| + |\eta|), \\ y - y_0 &= \frac{1+f_x^2}{\sqrt{(1+f_x^2+f_y^2)(1+f_x^2)}}\eta + o(|\xi| + |\eta|). \end{aligned}$$

Thus, from (7.2) and (7.5) we obtain

$$(7.6)\quad 2\zeta = A\xi^2 + 2B\xi\eta + C\eta^2 + o(\xi^2 + \eta^2),$$

where

$$\begin{aligned} A &= \frac{1}{(1+f_x^2+f_y^2)^{1/2}(1+f_x^2)} f_{xx}, \\ B &= \frac{1}{1+f_x^2+f_y^2}\left(f_{xy} - \frac{f_x f_y}{1+f_x^2} f_{xx}\right), \\ C &= \frac{1+f_x^2}{(1+f_x^2+f_y^2)^{3/2}}\left(f_{yy} - 2\frac{f_x f_y}{1+f_x^2} f_{xy} + \left(\frac{f_x f_y}{1+f_x^2}\right)^2 f_{xx}\right). \end{aligned}$$

We conclude that the osculating paraboloid is given by $2\zeta = A\xi^2 + 2B\xi\eta + C\eta^2$, and the invariants

$$\begin{aligned} K &= AC - B^2 = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}, \\ H &= \frac{A+C}{2} = \frac{(1+f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1+f_y^2) f_{xx}}{2(1+f_x^2+f_y^2)^{3/2}} \end{aligned}$$

are respectively the Gaussian curvature and the mean curvature of S at P.

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